

# Elliptic Orbits with a Non-Newtonian Eccentricity

F.T. Hioe\* and David Kuebel

Department of Physics, St. John Fisher College, Rochester, NY 14618,  
and

Department of Physics & Astronomy, University of Rochester, Rochester, NY 14627

August 2, 2012

## Abstract

It is shown that the lowest order general relativistic correction produces elliptic orbits with a non-Newtonian eccentricity.

PACS numbers: 04.20.Jb, 02.90.+p

In a weak gravitational field, the general relativistic effect of a massive object such as a star that produces a precessional motion [1] to an otherwise Newtonian elliptical orbit of a particle (such as a planet) is well known and well noted for its historical significance. The precessional angle  $\Delta\phi$  is approximately given by  $6\pi(GM/hc)^2$ , where  $M$  is the mass of the star,  $h$  is the angular momentum per unit rest mass of the particle,  $G$  is the universal gravitation constant, and  $c$  is the speed of light. Like the precessional angle of a particle in a weak gravitational field, for small  $s \equiv GM/hc$ , most lowest order general relativistic corrections are known to be of the order of  $s^2$  and higher. A general relativistic correction of the order  $s$ , on the other hand, is uncommon and is a principal result that we shall present in this Note. Specifically we shall present elliptic orbits with eccentricity given by  $\sqrt{2}s$ ; that is, we shall present a general relativistic effect of order  $s$  that makes circular Newtonian orbits elliptical, and the resulting elliptical orbits are non-precessing if terms of order  $s^2$  and higher can be neglected. In contrast, two new examples for hyperbolic orbits in which the lowest order general relativistic corrections are of the order  $s^2$  are also presented.

We start with one of the analytic solutions for the orbits in the Schwarzschild geometry that we presented in our papers [2-6]. We first introduce the parameters used in the analysis. The massive spherical object (which we call a star) of mass  $M$  sits at the origin of the coordinate system. Let the coordinates  $r$  and  $\phi$  describe the position of the particle relative to the star. If  $[x^\mu] = (t, r, \theta, \phi)$ , then the worldline  $x^\mu(\tau)$ , where  $\tau$  is the proper time along the path, of a particle moving in the equatorial plane  $\theta = \pi/2$ , satisfies the 'combined' energy equation [1]

$$\dot{r}^2 + \frac{h^2}{r^2} \left(1 - \frac{\alpha}{r}\right) - \frac{c^2 \alpha}{r} = c^2(\kappa^2 - 1), \quad (1)$$

where the derivative  $\dot{\phantom{x}}$  represents  $d/d\tau$ ,  $\alpha \equiv 2GM/c^2$  is the Schwarzschild radius,  $h = r^2 \dot{\phi}$  is identified as the angular momentum per unit rest mass of the particle, and the constant  $\kappa = E/(m_0 c^2)$  is identified to be the total energy per unit rest energy of the particle,  $E$  being the total energy of the particle in its orbit and  $m_0$  the rest mass of the particle at  $r = \infty$ .

By using the dimensionless distance  $q \equiv r/\alpha$  of the particle from the star measured in units of the Schwarzschild radius and another dimensionless quantity  $U$  defined by

$$U \equiv \frac{1}{4} \left( \frac{\alpha}{r} - \frac{1}{3} \right) = \frac{1}{4} \left( \frac{1}{q} - \frac{1}{3} \right), \quad (2)$$

eq.(1) reduces to the following simple form

$$\left( \frac{dU}{d\phi} \right)^2 = 4U^3 - g_2 U - g_3 \quad (3)$$

where

$$\begin{aligned} g_2 &= \frac{1}{12} - s^2 \\ g_3 &= \frac{1}{216} + \frac{1}{6}s^2 - \frac{1}{4}\kappa^2 s^2 \equiv \frac{1}{216} - \frac{1}{12}s^2 + \frac{1}{4}(1 - e^2)s^4, \end{aligned} \quad (4)$$

and where

$$s^2 \equiv \left( \frac{GM}{hc} \right)^2, \quad (5)$$

and

$$e^2 \equiv 1 + \frac{h^2 c^2 (\kappa^2 - 1)}{(GM)^2} \equiv 1 + \frac{\kappa^2 - 1}{s^2}. \quad (6)$$

The use of the dimensionless distance  $q$  led naturally to two dimensionless parameters  $\kappa^2$  (or  $e^2$ ) and  $s^2$  for characterizing the orbit. As was pointed in our previous work [2-6], the use of the parameter  $e^2$  makes the correspondence to the Newtonian case much easier to see. To demonstrate this, we use eqs.(6) and (1) to write

$$e^2 = \left( \frac{r^2 \dot{\phi}}{GM} \right)^2 \left\{ \dot{r}^2 + \left( r \dot{\phi} - \frac{GM}{r \dot{\phi}} \right)^2 - \frac{2GM}{c^2} r \dot{\phi}^2 \right\}, \quad (7)$$

and compare this expression with the Newtonian eccentricity  $e_N$ , which, using the derivative  $\dot{\cdot}$  to represent  $d/dt$ ,  $t$  being the ordinary time, can be expressed as

$$e_N^2 = \left( \frac{r^2 \dot{\phi}}{GM} \right)^2 \left\{ \dot{r}^2 + \left( r \dot{\phi} - \frac{GM}{r^2 \dot{\phi}} \right)^2 \right\}. \quad (8)$$

Comparing the two above expressions, it is seen that  $e \rightarrow e_N$  in the Newtonian limit implies the approximation  $\tau \rightarrow t$  and  $c \rightarrow \infty$ . However, since setting  $c = \infty$  is not consistent with reality, we will proceed by stating that  $e \rightarrow e_N$  if we take the approximation  $\tau \rightarrow t$  and

$$\dot{r}^2 + \left( r \dot{\phi} - \frac{GM}{r^2 \dot{\phi}} \right)^2 \gg \frac{2GM}{c^2} r \dot{\phi}. \quad (9)$$

We use the coordinates  $(e^2, s^2)$ , where  $-\infty \leq e^2 \leq +\infty$ ,  $0 \leq s \leq \infty$  of a parameter space for characterizing the two regions which we call Regions I and II for different types of orbits [5,6]. Region I is mathematically characterized by  $\Delta \leq 0$  and Region II is characterized by  $\Delta > 0$  where  $\Delta$  is the discriminant of the cubic equation

$$4U^3 - g_2U - g_3 = 0 \quad (10)$$

that is defined by

$$\Delta = 27g_3^2 - g_2^3 \quad (11)$$

and where  $g_2$  and  $g_3$  are defined by eq.(4). For the case  $\Delta \leq 0$ , the three roots of the cubic equation (10) are all real. We call the three roots  $e_1, e_2, e_3$  and arrange them so that  $e_1 > e_2 > e_3$ . In this paper, we are interested only in the orbit solution for which  $\Delta \leq 0$ ,  $e_1 > e_2 \geq U > e_3$  applicable in Region I. The equation for the orbit is [2,3]

$$\frac{1}{q} = \frac{1}{3} + 4e_3 + 4(e_2 - e_3)sn^2(\gamma\phi, k). \quad (12)$$

The constant  $\gamma$  appearing in the argument, and the modulus  $k$ , of the Jacobian elliptic functions [7] are given in terms of the three roots of the cubic equation (10) by

$$\gamma = (e_1 - e_3)^{1/2}, \quad (13)$$

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}. \quad (14)$$

where  $e_1, e_2, e_3$  are given by

$$\begin{aligned}
e_1 &= 2 \left( \frac{g_2}{12} \right)^{1/2} \cos \left( \frac{\theta}{3} \right), \\
e_2 &= 2 \left( \frac{g_2}{12} \right)^{1/2} \cos \left( \frac{\theta}{3} + \frac{4\pi}{3} \right), \\
e_3 &= 2 \left( \frac{g_2}{12} \right)^{1/2} \cos \left( \frac{\theta}{3} + \frac{2\pi}{3} \right),
\end{aligned} \tag{15}$$

and where

$$\cos \theta = g_3 \left( \frac{27}{g_2^3} \right)^{1/2}. \tag{16}$$

A typical orbit given by eq.(12) (not on any one of the three boundaries) in Region I is a precessional elliptic-type orbit for  $e^2 < 1$ , a parabolic-type orbit for  $e^2 = 1$ , and a hyperbolic-type orbit for  $e^2 > 1$  [5,6].

For the elliptic-type orbits ( $e^2 < 1$ ), the maximum distance  $r_{\max}$  (the aphe-  
lion) of the particle from the star and the minimum distance  $r_{\min}$  (the perihelion)  
of the particle from the star, or their corresponding dimensionless forms  $q_{\max}$   
( $= r_{\max}/\alpha$ ) and  $q_{\min}$  ( $= r_{\min}/\alpha$ ), are obtained from eq.(12) when  $\gamma\phi = 0$  and  
when  $\gamma\phi = K(k)$  respectively, where  $K(k)$  is the complete elliptic integral of  
the first kind [7], and they are given by

$$\frac{1}{q_{\max}} = \frac{1}{3} + 4e_3, \tag{17}$$

and

$$\frac{1}{q_{\min}} = \frac{1}{3} + 4e_2. \tag{18}$$

The geometric eccentricity  $\varepsilon$  of the orbit is defined in the range  $0 \leq \varepsilon \leq 1$   
by

$$\varepsilon \equiv \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = \frac{q_{\max} - q_{\min}}{q_{\max} + q_{\min}} = \frac{e_2 - e_3}{1/6 + e_2 + e_3}, \tag{19}$$

using  $q_{\max}$  and  $q_{\min}$  given by eqs.(17) and (18). It has been shown in ref.3  
that in the range  $0 \leq \varepsilon < 1$  that  $\varepsilon \rightarrow e$  from below as  $s \rightarrow 0$ , and that  $\varepsilon = e$  for  
all values of  $s$  when  $\varepsilon = 1$ .

The precessional angle  $\Delta\phi$  is given by

$$\Delta\phi = \frac{2K(k)}{\gamma} - 2\pi. \tag{20}$$

The Newtonian correspondence is approached by making  $s$  very small. Sub-  
stituting eq.(4) into eq.(16) and expanding in power series in  $s$ , we find

$$\cos \theta = 1 - 2 \cdot 3^3 e^2 s^4 - 2^2 \cdot 3^3 (1 + 9e^2) s^6 - 2 \cdot 3^5 \cdot 5 (1 + 6e^2) s^8 + \dots \tag{21}$$

To obtain a power series in  $s$  for  $\theta$ , the point  $e^2 = 0$  must be done separately. For  $e^2 > 0$ , we have

$$\theta = 2 \cdot 3\sqrt{3}es^2 [1 + e^{-2}(1 + 9e^2)s^2 + \dots] . \quad (22)$$

Expanding  $e_1, e_2, e_3, \gamma, k^2, sn(\gamma\phi, k)$  in eq.(12),  $\varepsilon$  in eq.(19), and  $\Delta\phi$  in eq.(20) in the power series in  $s$ , we find that the orbit equation (12) can be approximated for  $e^2 > 0$  and for small  $s^2$  by

$$\frac{1}{q} = 2s^2 \{1 - \varepsilon \cos[(1 - \delta)\phi]\}, \quad (23)$$

which, in terms of  $r$ , gives the approximate orbit equation

$$\frac{1}{r} = \frac{GM}{h^2} \{1 - \varepsilon \cos[(1 - \delta)\phi]\}, \quad (24)$$

where  $\varepsilon$ , to the order of  $s^2$ , is given by

$$\varepsilon \simeq e + (e^{-1} - e^3)s^2, \quad (25)$$

and where  $\delta$ , to the order of  $s^2$ , is given by

$$\delta \simeq 3s^2. \quad (26)$$

$\delta$  is related to the precessional angle  $\Delta\phi$  given in eq.(20) by  $\Delta\phi \simeq 2\pi\delta \simeq 6\pi s^2 = 6\pi[GM/(hc)]^2$  and it is independent of  $e$  (to the order  $s^2$ ) for  $0 < e \leq \infty$ . As an example, a general relativistic elliptic-type orbit with  $e = 0.8$ ,  $s = 0.0176539$  has an exact  $\varepsilon = 0.80015$ . The lowest order general relativistic corrections (25) and (26) yield  $\varepsilon \simeq 0.80023$  and  $\delta \simeq 0.000935$ . The corrections to  $\varepsilon \simeq e$  and the magnitude of  $\delta$  are both of the order  $s^2$ .

Thus if we can ignore terms of order  $s^2$  and higher, we recover the Newtonian orbit equation given by

$$\frac{1}{r} = \frac{GM}{h^2} (1 - e \cos \phi). \quad (27)$$

The question that can be posed at this point is whether we can define the Newtonian limit by stating that it is the general relativistic result for small  $s$  if we ignore terms of order  $s^2$  and higher.

To proceed, we note that the case  $e = 0$  is excluded from the expansion given by eq.(22) and it is also clear that  $e^2 = 0$  does not satisfy the condition given by eq.(9) and must be treated separately.

For  $e^2 = 0$ , the expansion for  $\theta$ , instead of eq.(22), is now

$$\theta = \sqrt{2^3 \cdot 3^3} s^3 \left( 1 + \frac{3^2 \cdot 5}{2^2} s^2 + \dots \right) \quad (28)$$

and the approximate orbit equation still has the form of eq.(23) or (24) with the same  $\delta$  given by eq.(26) to the order of  $s^2$ , but with  $\varepsilon$ , instead of eq.(25), now given to the order of  $s^3$  by

$$\varepsilon = \sqrt{2}s \left(1 + \frac{9}{4}s^2\right). \quad (29)$$

Thus we have elliptic orbits that precess with the same angle  $\delta$  given by eq.(26) but with an eccentricity equal to  $\sqrt{2}s + 9\sqrt{2}s^3/4$  to the order  $s^3$ . If we ignore terms of order  $s^2$  and higher, the orbit equation becomes

$$\frac{1}{r} = \frac{GM}{h^2}(1 - \sqrt{2}s \cos \phi), \quad (30)$$

which is a (non-precessing) elliptical orbit with a non-Newtonian eccentricity that is dependent on the speed of light. To the best of our knowledge, this general relativistic elliptical orbit with an eccentricity of the order  $s$  is new and has never been noted by other authors. As an example, an elliptic orbit with eccentricity  $\varepsilon = 0.0368035$  could come from a general relativistic orbit with  $e = 0$  [remembering that  $e$  is defined by eq.(7) and not eq.(8)] and  $s = 0.0259843$  for which the approximation formula (29) gives  $\varepsilon \simeq 0.0368032$  [the first term alone gives  $\sqrt{2}s = 0.0367474$ ] and eq.(26) gives  $\delta \simeq 0.0020256$ . The approximation  $\varepsilon \simeq e = 0$  holds if we ignore terms of order  $s$ . The orbit for  $e = 0$  from general relativity becomes a Newtonian circular orbit if we ignore terms of order  $s$ , i.e. ignoring the second order terms in  $s^2$  and higher order terms is not sufficient to get the Newtonian limit for this case.

It is clear that the entire region characterized by  $e^2 \leq 0$  or

$$\dot{r}^2 + \left(r\dot{\phi} - \frac{GM}{r^2\dot{\phi}}\right)^2 \leq \frac{2GM}{c^2}r\dot{\phi}^2 \quad (31)$$

is non-Newtonian in character. This includes all circular orbits that occur [6] on the curve  $s^2 = s_1'^2$  for which  $k^2 = 0$  and  $\varepsilon = 0$ , where  $s_1'^2$  is given by

$$s_1'^2 = \frac{1 - 9e^2 - \sqrt{(1 + 3e^2)^3}}{27(1 - e^2)^2} \quad (32)$$

from the "vertex"  $V$  at  $(e^2, s^2) = (-1/3, 1/12)$  where the innermost stable circular orbit (ISCO) occurs, to the origin  $O$  at  $(e^2, s^2) = (0, 0)$  where the circular orbit has an infinite radius. This curve  $s_1'$  defines a boundary of Region I for which the values of  $e^2$  range between  $-1/3$  and  $0$  and the values for  $s^2$  range between  $1/12$  and  $0$ . All circular orbits precess even though the precession angle is not observable [8], and for small  $s$  the precession angle is given by

$$\Delta\phi \simeq 6\pi s^2 \simeq 6\pi \frac{GM}{c^2 r_c},$$

where  $r_c \simeq h^2/(GM)$  is the radius of the circular orbit, and  $\Delta\phi$  is non-zero unless the radius of the circle is infinite which occurs on  $s = 0$  for zero gravitational field.

The values of  $e^2$  along the  $s_1'$  curve where the circular orbits occur near  $s = 0$  are given by

$$e^2 \simeq -2s_1'^2.$$

For small  $s_1'$  and for  $s$  just above  $s_1'$  inside Region I given by  $s^2 = s_1'^2 + (\Delta s)^2$ , it can be shown in the same manner that to the order  $\Delta s$  we have elliptic orbits similar to eq.(30) given by

$$\frac{1}{r} = \frac{GM}{h^2} [1 - \sqrt{2}(\Delta s) \cos \phi]. \quad (33)$$

For the parabolic-type orbit ( $e^2 = 1$ ),  $e_3 = -1/12$  and the initial distance of the particle from the star is given from eq.(17) to be  $q_{\max} = \infty$  and eq.(19) gives  $\varepsilon = 1$ . Thus  $e = 1$  and  $\varepsilon = 1$  coincide for all values of  $s$ . The orbit equation is given exactly by

$$\frac{1}{q} = 4(e_2 + \frac{1}{12})sn^2(\gamma\phi, k), \quad (34)$$

and for small  $s$  values approximately by

$$\frac{1}{r} = \frac{GM}{h^2} \{1 - \cos[(1 - \delta)\phi]\}, \quad (35)$$

where  $\delta$  is given by eq.(26) and for which the lowest order general relativistic correction to the Newtonian case is of the order  $s^2$ .

For the hyperbolic-type orbit ( $e^2 > 1$ ),  $e_3$  is less than  $-1/12$  and eq.(17) is not applicable. Instead, a particle approaches the star from infinity along an incoming asymptote at an angle  $\Psi_1$  to the horizontal axis given by [2,3]

$$\Psi_1 = \gamma^{-1}sn^{-1} \left( \sqrt{-\frac{\frac{1}{3} + 4e_3}{4(e_2 - e_3)}}, k \right), \quad (36)$$

where  $\gamma$  and  $k$  are defined by eqs.(13) and (14), turns counter-clockwise about the star to its right on the horizontal axis, and leaves along an outgoing asymptote at an angle  $\Psi_2$  given by

$$\Psi_2 \equiv \frac{2K(k)}{\gamma} - \Psi_1. \quad (37)$$

The minimum dimensionless distance  $q_{\min}$  of the particle from the star is still given by eq.(18) as  $e_2$  is still greater than  $-1/12$ .

In the Newtonian limit for small  $s$ ,  $\Psi_1$  becomes

$$\Psi_1 \simeq \cos^{-1}(\frac{1}{e}) \equiv \phi_0, \quad (38)$$

and the complementary angle  $\Psi_1' \equiv 2\pi - \Psi_2 \simeq \phi_0$  also. If we define

$$\Theta_{GR} \equiv \Psi_1 + \Psi_1' \quad (39)$$

and

$$\Theta_{Newton} = 2\phi_0, \quad (40)$$

the difference  $\Delta\phi \equiv \Theta_{Newton} - \Theta_{GR}$  can be taken to be an analog of the precession angle given by eq.(20) for a hyperbolic orbit, and for small  $s$  and to the order of  $s^2$ , it was shown [3] to be given by

$$\Delta\phi \simeq \left[ 6\pi - 6\phi_0 + 2(2 + e^{-2})\sqrt{e^2 - 1} \right] s^2. \quad (41)$$

This result is different from the approximation used by Longuski et al. [9]. As discussed in ref.9, an experimental test can be carried out to check this result.

From eq.(18), the minimum distance  $r_{\min}$  of the particle from the star is given approximately by

$$\frac{1}{r_{\min}} \simeq \frac{GM}{h^2} (e + 1) \left[ 1 + \frac{(e + 1)^2}{e} s^2 + \dots \right]. \quad (42)$$

Equations (41) and (42) show two examples for which the lowest general relativistic corrections to the Newtonian case are of the order  $s^2$  and not  $s$  for  $e^2 > 1$ .

To summarize the above results, for small values of  $s$ , the general relativistic correction to the Newtonian elliptic orbit is second order in  $s$  for  $e^2 > 0$  but is first order in  $s$  for  $e^2 \leq 0$  for which the correction is shown to appear in the eccentricity of the elliptic orbit. This division gives a new meaning and significance to the parameter  $e^2$ . In particular, there exist non-Newtonian elliptic orbits of eccentricity  $\sqrt{2}s$  given by eq.(30).

## References

\*Electronic address: fhioe@sjfc.edu

- [1] M.P. Hobson, G. Efstathiou and A.N. Lasenby: General Relativity, Cambridge University Press, 2006, Chapters 9 and 10.
- [2] F.T. Hioe, Phys. Lett. A 373, 1506 (2009).
- [3] F.T. Hioe and D. Kuebel, Phys. Rev. D 81, 084017 (2010).
- [4] F.T. Hioe and D. Kuebel, arXiv:1008.1964 v1 (2010).
- [5] F.T. Hioe and D. Kuebel, arXiv:1010.0996 v2 (2010).
- [6] F.T. Hioe and D. Kuebel, arXiv:1207.7041v1 (2012).
- [7] P.F. Byrd and M.D. Friedman: Handbook of Elliptic Integrals for Engineers and Scientists, 2nd Edition, Springer-Verlag, New York, 1971.
- [8] J.L. Martin: General Relativity, Revised Edition, Prentice Hall, New York 1996, Chapter 4.
- [9] J.M. Longuski, E. Fischbach, and D.J. Scheeres, Phys. Rev. Lett. 86, 2942 (2001), J.M. Longuski, E. Fischbach, D.J. Scheeres, G. Giampierri, and R.S. Park, Phys. Rev. D 69, 042001 (2004).